Bank Shot

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Purpose	Students compare their own reasoning strategies and those of their classmates, focusing on the strategies' usefulness in determining how to make certain bank shots in billiards. This task is intended to involve multiple geometric perspectives and would be appropriate for students with an understanding of similar triangles, rigid motions (reflections), and equations for lines and is designed to develop students' understanding of these concepts.	
Task Overview	Given a cue ball in a particular position, determine where the ball should strike a particular side of the billiard table on a path to a given corner of the playing surface. Justify your solution. *An activity sheet that gives students the complete task is included.*	
Focus on Reasoning and Sense Making	**Reasoning Habits** ***Focus in High School Mathematics: Reasoning and Sense Making*** Analyzing a problem—seeking relationships; looking for structure Implementing a strategy—making logical deductions Reflecting on a solution—justifying a solution; reconciling different approaches **Process Standards** ***Principles and Standards for School Mathematics*** Problem Solving—apply and adapt appropriate strategies to solve problems Reasoning and Proof—develop and evaluate mathematical arguments and proofs Communication—communicate mathematical thinking clearly; analyze and evaluate the mathematical thinking and strategies of others Connections—understand how mathematical ideas interconnect	**Standards for Mathematical Practice** **Common Core State Standards for Mathematics** 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 7. Look for and make use of structure.
Focus on Mathematical Content	**Key Elements** ***Focus in High School Mathematics: Reasoning and Sense Making*** Reasoning with geometry—construction and evaluation of geometric arguments; multiple geometric approaches	**Standards for Mathematical Content** **Common Core State Standards for Mathematics** G-CO-6. Use geometric descriptions of rigid motions to … predict the effect of a given rigid motion on a given figure. G-CO-9. Prove theorems about lines and angles. G-SRT-5. Use congruence and similarity criteria for triangles to solve problems.
Materials and Technology	Bank Shot activity sheet	

Use in the Classroom

Before having the students begin the activity, you might talk about billiards or show a brief video to familiarize students with the setting of the problem. (A search for “carom billiards” will reveal many online sources.)

Distribute page 1 of the activity sheet, read the introduction and question 1 (or ask students to read them), and have the students work on question 1 in groups. (Wait until after the students have discussed their work on question 1 before distributing page 2 of the activity sheet.)

In addition to having students answer the questions on the activity sheet, the objective is to have them create multiple solution strategies and compare, critique, and reflect on them. Working in groups may elicit different approaches that the students can discuss and pursue.

Students in groups are encouraged to pursue multiple strategies.

If a group is having trouble getting started, you might ask questions such as the following:

Analyzing structure and relationships is a good way to move forward.

- “What geometric structures do you see here?”
- “What mathematical concepts come to mind?”
- “How are parts of the figure related to one another?”
- “What additional structure or relationships can you see or create?”
- “What pieces of given information are useful?”

When the students have completed work on question 1, you might have groups present their solutions and invite comments from their classmates. Try to elicit for discussion at least two different successful and justified strategies—preferably using different aspects of geometry. (The discussion in the “Focus on Student Thinking” section provides some possibilities.) If your students suggest only one type of solution strategy, facilitate their thinking about another approach—without giving too much detail. For instance, if students initially offer only solutions involving similar triangles, you might prompt them to search for a solution that involves transformational geometry by saying something like, “I seem to remember somebody using reflections of the ball’s path somehow. Can you think of a solution strategy that uses reflections?” (See the “Focus on Student Thinking” section for details.)

Students have two roles: explaining the reasoning behind their solutions and critiquing the reasoning of classmates.

It is important for you to lead a discussion that engages students in comparing and contrasting solution strategies. This discussion might include, but would not be limited to, the strengths and weaknesses of different approaches.

Having students reflect on solutions can have benefits beyond particular problems.

Next, distribute page 2 of the activity sheet and have your students work on question 2. Most probably, they will try to generalize (at least two) of the strategies from the discussion of question 1. If they have not already done so, help them express some important general observations, noting, for instance, that different solution strategies might reveal different aspects of a problem and that some solution strategies might be more easily extended to certain groups of problems. (See the “Focus on Student Thinking” section for an example of where this might be the case.)

Multiple approaches can contribute to sense making.

As time permits, you might have students explore question 3, which varies the original task in a way that allows students to adapt a strategy or use a new approach.

Focus on Student Thinking

Multiple strategies lead to solutions in this activity. Some groups might approach question 1 by using similar triangles (see fig. A). Without needing to use the actual length of the table, a group might obtain and justify a relative answer, such as, “You need to hit the cue ball so that it hits the cushion at a point that is one-third of the distance down the remaining length of the table.” Another group might use all the given measurements and conclude that the cue ball should hit the side of the table 5 1/3 feet from the end of the table.

A different group might decide merely to depend on figure 2 on the activity sheet and what it seems to show—namely, that “you need to hit the cue ball so that the two segments of the ball’s path form a right angle.” Actually, the angle is slightly less than 94 degrees, and in critiquing this strategy, students might find many ways to show that the conclusion about a right angle is wrong. In fact, the cue ball would miss the corner by more than 6 inches if students were to implement this strategy!

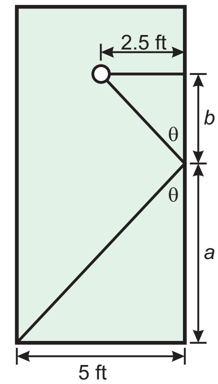

Fig. A

To introduce into the discussion a solution with a different approach, you might facilitate a search for a strategy that involves transformational geometry. One group might decide to reflect the playing surface and the lower portion of the cue ball’s path across the cushion edge, as shown in figure B. Then, using the fact that a reflected angle retains its original measure, the students might use elementary results in Euclidean geometry, along with the given assumptions, to prove a relationship about the resulting configuration—namely, that the upper segment of the ball’s path and the reflected lower segment of its path lie on a straight line.

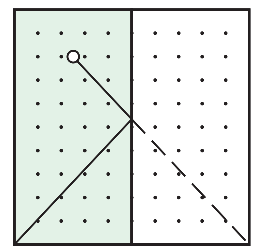

Fig. B

Having reached this point, the group might shift to the use of coordinate geometry. By placing the figure on a coordinate grid (as in fig. C), they can derive a linear equation for this line, since they can easily determine the coordinates of two points on this line from the given information. Then they can use this equation to determine the correct point of contact between the cue ball the cushion on the table.

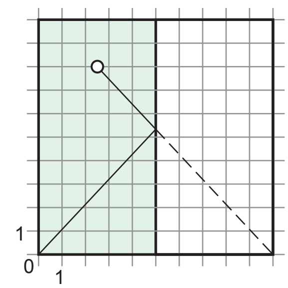

Fig. C

Another group might use a reflection in a different way. They might place the table on a grid first and then construct a line of reflection that is perpendicular to the right cushion edge of the table at a (potential) point of contact between the cue ball and the right cushion (see fig. D). Using properties of a reflection, they might show that the line containing the upper segment of the ball’s path is the reflected image of the line containing the lower segment of its path. Because each one of these lines contains a point whose coordinates are known, the next step for the students might be to find the slope of each line so that they could write equations for the two lines. Then, using these equations, they could determine where the two lines intersect and, thus, solve the problem. Thinking conceptually of slope as the change in y divided by the corresponding change in x, the students might use properties of the reflection to conclude that as x increases from, say, 2 1/2 to 5, the rise in y- values associated with the lower line of the ball’s path is equal to the decrease in y-values associated with the upper line of the ball’s path. Therefore, the slope of the upper line is the negative of the slope of the lower line. (Students may already know this fact from previous study. If not, it might be worth developing a proof of this abstract result during the discussion of student solutions.) This connection between the slopes of the two lines of the ball’s path, together with the fact that the two lines meet when $x = 5$, provides students with enough additional information to determine the actual value of the slope of each line and, subsequently, the point of contact between the cue ball and the right cushion.

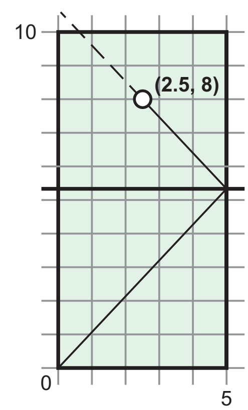

Fig. D

Focus on Student Thinking—*Continued*

Note that we have discussed only some of the strategies that are possible. Assuming that your students present all of the strategies mentioned above, a discussion comparing them before working on question 2 can help students reflect on their intricacies, advantages, and disadvantages. For instance, during such a discussion, students might suggest that the very first answer given above, which is relative to the table's length, might provide a good approach for someone ready to strike a real cue ball at the given position. Some students might suggest altering the first reflection strategy so that it uses similar triangles instead of imposing a grid and determining the equation of a line. The second reflection strategy might appeal to students who immediately decided that they would solve the problem by finding equations of the two lines related to the ball's path, but got stuck. Finally, this may be a time to document more general results, questions, or conjectures that arise from this mathematical reasoning activity. (See the discussion of the second reflection strategy above.)

With differing degrees of difficulty, each of the solution strategies identified for question 1 can be extended to provide a solution for question 2. The strategies using similar triangles now require comparing more than two proportions. The first reflection strategy requires more than one reflection to "straighten out" the bank shot into a single line. The second reflection strategy involves determining and solving interrelated pairs of linear equations. Some students might be able to outline the basic elements of a general strategy but be unable to succeed in implementing it. Outlining a strategy is a beginning. At the other extreme, students who are successful with multiple strategies might prefer one strategy or see one strategy as more valuable than another. For example, some students might suggest that the first reflection strategy provides an "easier" way to analyze more complicated situations than other solutions presented. It might be worth asking for a precise definition of *easier*. Does *easier* mean shorter, clearer, more generalizable, less complex, or what exactly?

In answering question 3, students might notice (and prove) that the complete path of the cue ball contains the sides of an isosceles triangle. They might then proceed to use congruent triangles to describe a solution. You might ask, "Which solution would you recommend, and why?"

Assessment

To gain insight into students' reasoning, you might give a homework assignment asking students to describe, implement, and critique two strategies that they could use to solve another problem about the path of a cue ball. For example, one such problem might involve striking the cue ball so that it banks off the lower cushion and then travels to the top right-hand corner.

Extensions

Many extensions are possible—for example, students can invent and justify complicated shots like a five-cushion bank shot that ends in a corner, different shots that eventually "loop" around the same path, or shots that would never hit a corner or loop.

Resources

Common Core State Standards Initiative (CCSSI). *Common Core State Standards for Mathematics. Common Core State Standards (College- and Career-Readiness Standards and K–12 Standards in English Language Arts and Math)*. Washington, D.C.: National Governors Association Center for Best Practices and the Council of Chief State School Officers, 2010. http://www.corestandards.org.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

———. *Focus in High School Mathematics: Reasoning and Sense Making.* Reston, Va.: NCTM, 2009. Example 13, pp. 52–53.

Bank Shot

Student Activity Sheet

Carom billiards refers to a collection of games typically played on a 5-by-10-foot “pocketless” rectangular table (see fig. 1). In this game, a player scores points by striking a *cue ball* with a cue stick so that the cue ball hits each of two additional billiard balls on the table before the cue ball comes to rest. Some shots may require striking the cue ball so that it hits a cushion (the side of the table) during the shot. Such a shot is called a *bank shot*. Angles matter in these games! In this activity, you will investigate some trajectories of the cue ball when it is alone on the table.

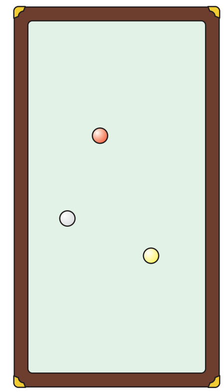

Fig. 1

1. Assume that a single cue ball is placed exactly halfway across the width of the table and two feet from its top end. Your task is to determine where the cue ball should strike the cushion on the long side of the billiard table so that the ball would ricochet and be on a path to hit the lower left-hand corner of the table (see fig. 2). You can assume that whenever the cue ball hits a side cushion, it bounces off in such a way that the angle made by the table’s side cushion and the path of the ball as it hits the cushion is the same as the angle made by the table’s side and the path of the ball as it bounces off the cushion. That is, as is typical of many billiard shots, the *angle of incidence* equals the *angle of reflection*, as labeled in figure 2.

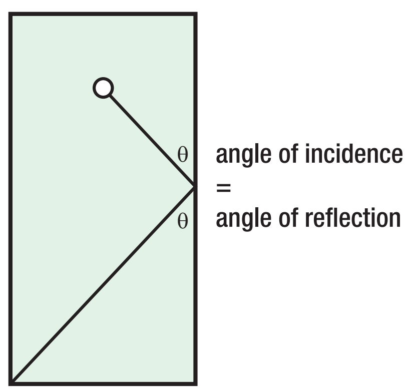

Fig. 2

Try to justify your answer in as many ways as possible and be ready to discuss your solutions.

2. On the same billiard table and with the same initial ball placement and same assumptions as in question 1, determine where the ball should strike the long side of the table so that a two-cushion bank shot will put the ball on a path that ends at the upper left-hand corner (see fig. 3).

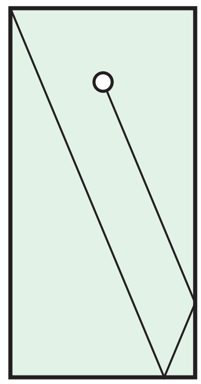

Fig. 3

Try at least two different solution strategies—even if your first strategy is successful. Do you prefer one strategy to the other? Do you think you could use either or both of your two strategies with three or more cushion bank shots? Explain.

3. Suppose that an additional billiard ball is now placed in the center of the table (see fig. 4). The cue ball is in its original position. Rather than hit a corner, the objective now is to determine a bank shot against the right cushion that will result in the cue ball hitting the new billiard ball after it hits the cushion.

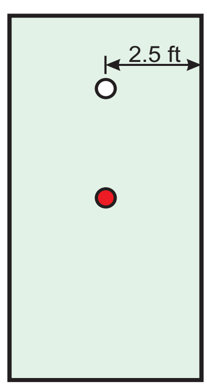

Fig. 4